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Phase diagram and correlation length bounds for Mandelbrot aerogels

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Abstract. We study a variant of the Mandelbrot percolation process which is of current use as a model of aerogels. The model has two parameters: One of them, Q, is the usual multiscale parameter of the Mandelbrot percolation process and the other, p, is a Bernoulli percolation parameter that is reserved for 'the last step of the construction'. We investigate the phase diagram of this model in the (Q,p) plane. There are two phases, a sol phase and a gel phase, classified according to whether a limit of crossing probabilities vanishes or is non-zero. In the sol phase, we define a correlation length via the rate of decay of a rescaled connectivity function. We show that this length scale diverges at the phase boundary. Furthermore, we demonstrate that if the phase boundary is approached with Q fixed and p tending up to its critical value, $P_G(Q)$, then, up to logarithmic corrections, the divergence is at least as fast as $|P_G-p|^{-2/d_H}$, where $d_H=2-|\log Q/\log N|$ can be identified as the Hausdorff dimension of the background medium.

1. Introduction and statement of results

In this paper, we establish some basic properties of the phase diagram and correlation length for a variant of the Mandelbrot percolation process which we call the *Mandelbrot aerogel*. Here we will be concerned exclusively with the two-dimensional case.

Aerogels are porous materials with pore sizes ranging from nanometers to microns [1, 2]. (See [3] and [4] for reviews.) These materials seem to have a long-range-correlated, fractal type of randomness over at least two decades of length scales [5], as well as independent randomness on a small scale. Motivated by this structure, a two-parameter Mandelbrot percolation model of aerogels was introduced [6]. This model may be loosely described as a Bernouilli percolation problem on the random sets generated in a Mandelbrot percolation process.

Models of aerogels are of current interest due to the use of aerogels in a broad range of practical applications, e.g. insulating devices and solar collectors [4], as well as to theoretical questions raised by the recent experiments on the behaviour of superfluid helium in aerogel glasses [7–9].

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Let us begin with a brief review of the standard Mandelbrot percolation process described in [10]; for simplicity of exposition, we will here confine attention to the two-dimensional case. The process is defined on the unit square $[0,1]^2$ which we denote by A_0. At the first stage, A_0 is subdivided into N^2 smaller squares (cells)

$$S_{i,k} = \left[\frac{i-1}{N}, \frac{i}{N}\right] \times \left[\frac{k-1}{N}, \frac{k}{N}\right] \qquad 1 \leqslant i, k \leqslant N$$

with $N \geqslant 2$. The smaller squares $S_{i,k}$ are independently retained (occupied) with probability Q or discarded (vacated) with probability $(1-Q)$. The closure of the collection of retained squares constitutes the set A_1. The set $A_2 \subset A_1$ is generated by repeating the previous construction (appropriately scaled) on all the surviving squares of A_1. The process is iterated, so that at the nth stage, the squares being retained and discarded are of the form

$$\left[\frac{i-1}{N^n}, \frac{i}{N^n}\right] \times \left[\frac{k-1}{N^n}, \frac{k}{N^n}\right] \qquad 1 \leqslant i, k \leqslant N^n$$

and evidently are of scale N^{-n}. The subject of interest is the limiting set

$$A_\infty = \bigcap_n A_n. \tag{1.1}$$

Most of the rigorous work on Mandelbrot percolation concerns the connectivity properties of this set.

The phase structure and various other properties of the two-dimensional process were studied in [11]. It was shown that, for fixed $N \geqslant 2$, as a function of Q, the limiting set undergoes several transitions, the most interesting of which is a percolation-like transition at some non-trivial $Q_c(N)$. This transition is signalled by the vanishing or non-vanishing of the 'crossing probability'

$$\theta_\infty(Q) = \text{Prob}(A_\infty \text{ contains a path connecting the left and right sides of } [0,1]^2). \tag{1.2}$$

In particular, for $Q < Q_c$, $\theta_\infty(Q)$ is zero, while for $Q \geqslant Q_c$, $\theta_\infty(Q)$ is (strictly) positive. It was also established in [11] that, whenever there is positive probability that $A_\infty \neq \varnothing$, with conditional probability equal to one, the Hausdorff dimension of A_∞ is the naively anticipated: $d_H = 2 - |\log Q/\log N|$.

The Mandelbrot aerogel, at the nth stage, is defined by first running $n-1$ iterations of the Mandelbrot process with retention parameter Q (thereby obtaining some random set A_{n-1}), and then doing the nth iteration at retention parameter p. Of course this is equivalent to doing ordinary Bernoulli percolation at density p on the random subset of the $N^n \times N^n$ grid contained within A_{n-1}.

Unfortunately there is, in general, no obvious way to arrive at a limiting set for the Mandelbrot aerogel. Therefore, in this paper, we will be content to make probabilistic statements about the nth aerogels that hold uniformly in n.

As usual, the questions of interest focus on the connectivity properties of the sets A_n. (In this work, we *define* neighbouring cells to be cells which have an edge in common; this definition underlies the relevant notion of connectedness.) Of central importance will be the crossing probability:

$$\theta_n(Q,p) = \text{Prob}(A_n \text{ contains an occupied path connecting the left and right sides of } [0,1]^2) \tag{1.3}$$

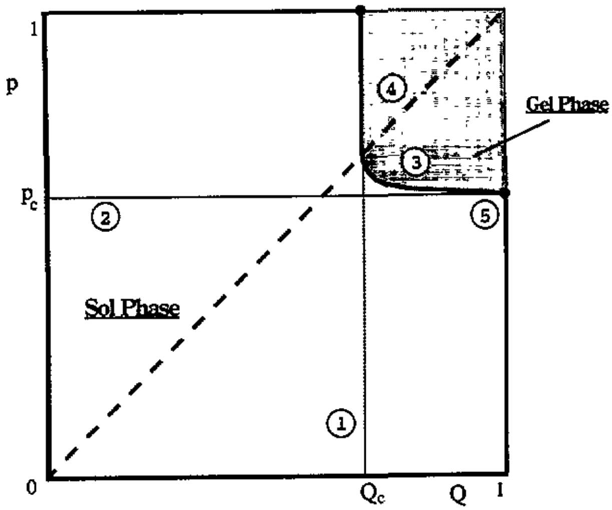


Figure 1. Phase diagram of the Mandelbrot aerogel.

where in the above (and henceforth whenever possible) we suppress all N dependence in our notation. Although we cannot, in general, assure the existence of an $n \to \infty$ limit for $\theta_n(Q,p)$, we can always define

$$\underline{\theta}(Q,p) = \liminf_{n\to\infty} \theta_n(Q,p). \tag{1.4}$$

For those (Q,p) where it can be demonstrated that a limit actually exists—which turns out to be when $\underline{\theta}(Q,p)=0$ and/or when $p \geqslant Q$—we will denote this limit by $\theta(Q,p)$. We will declare the system to be in the gel phase if $\underline{\theta}(Q,p)>0$; otherwise the system is an aerosol†.

The results in section 1 concern the phase diagram of the Mandelbrot aerogels as a function of p and Q. We first show, in proposition 2.1, that throughout the sol phase, the $\theta_n(Q,p)$ actually converge to zero. As a corollary to proposition 2.1, we demonstrate that $\underline{\theta}(Q,p)$ is discontinuous across the phase boundary. This is analogous to the situation in the ordinary Mandelbrot percolation process. In theorem 1.1, we describe most of the essential features of the phase diagram in terms of the transition thresholds, p_c and Q_c, of the ordinary Bernoulli (site) and Mandelbrot percolation models. These features are illustrated in figure 1 (where the numbers in the figure correspond to the parts of theorem 1.1). For convenience, we state this theorem below, in its entirety.

Theorem 1.1. Consider the (Q,p)-Mandelbot aerogel model with fixed parameter N:
(1) If $Q<Q_c$, the model is in the sol phase ($\theta(Q,p)=0$).
(2) If $p \leqslant p_c$ and $Q<1$, the model is in the sol phase ($\theta(Q, p)=0$).
(3) If $p>p_c$, there exists $Q<1$ so that the model is in the gel phase $\underline{\theta}(Q,p)>0$).

† It should be noted that although $\underline{\theta}(Q,p)$ distinguishes the disordered (sol) phase from the ordered (gel) phase by being non-zero only in the latter, it is almost certainly not an order parameter for this system. In particular, one would be hard pressed to produce a conjugate (ordering) field which couples linearly to $\underline{\theta}(Q,p)$.

(4) *If* $p \geqslant Q$, the limit of the crossing probabilities $(\theta_n(Q,p))$ exists and is equal to the crossing probability $\theta_\infty(Q)$ of the ordinary Mandelbrot process at retention parameter Q. In particular, if $p \geqslant Q \geqslant Q_c$, the model is in the gel phase ($\theta(Q,p) > 0$), so that the vertical line from (Q_c, Q_c) to $(Q_c, 1)$ is part of the phase boundary.

(5) For a $Q > Q_c$, the phase boundary may be expressed as a monotone non-increasing function, $p = P_G(Q)$, with $P_G(1) = p_c$.

In section 2, we define two correlation lengths for the sol phase. Our first length, $\xi_1(Q,p)$, is a finite-size scaling correlation length of the sort described in [12]. Here we use the smallest scale where vacant cracks separating opposing sides of boxes are typically observed. It turns out that the proper units in which this length should be expressed are the microscopic units of the individual cells. Thus, if cracks 'typically apear' in A_{n^*}, the finite-size length is N^{n^*}. By fiat, this length scale diverges as the sol–gel phase boundary is approached. The second correlation length is defined via the connectivity function: if a and $a+(s,0)$ are points in $[0,1]^2$, we denote by $\tau_{a;s}^{[n]}$ the probability that, in the set A_n, these points are part of the same connected cluster. In the sol phase, as $n \to \infty$, $\tau_{a;s}^{[n]}$ tends to zero exponentially fast in the distance (as measured in microscopic units) between the two points. Thus we may write

$$\tau_{a;s}^{[n]} \sim \exp[-sN^n/\xi_2] \tag{1.5}$$

where the existence of the limit implicit in equation (1.5) is established in lemma 3.1. Demonstrating the existence of ξ_2 in fact constitutes the bulk of our efforts for this section; most of the proof of lemma 3.1 should be skimmed in a preliminary reading. We conclude section 2 with theorem 3.5 which is a straightforward demonstration of the 'scaling equivalence' of ξ_1 and ξ_2. In particular, if we are in the sol phase at the point (Q,p) with $Q \geqslant Q_c$, and one of these lengths, ξ, exhibits critical scaling in the sense that the limit ν 'defined' by

$$\nu = \lim_{p \uparrow P_G(Q)} \left| \frac{\log \xi(Q,p)}{\log|P_G - p|} \right| \tag{1.6}$$

exists, then a similar limit also exists and also equals ν for the other length.

The results of section 3 concern the critical behaviour of the correlation length for the Mandelbrot aerogels. In [12] it was shown that in a d-dimensional system with independent disorder characterized by a (percolation-like) density parameter p, any appropriately defined finite-size scaling correlation length, ξ_f, which diverges as p approaches a critical p_c obeys the bound

$$\xi_f \geqslant \text{constant}\,|p - p_c|^{-2/d} \tag{1.7a}$$

uniformly in a neighbourhood of p_c. Assuming that $\xi_f(p)$ actually tends to infinity with a power law, $\xi_f(p) \sim |p-p_c|^{-\nu_f}$, this implies the bound $\nu_f \geqslant 2/d$ on the correlation length exponent. For percolation [13, 14] and disordered Ising ferromagnets [15] it was shown that, in the scaling sense, a finite-size scaling correlation length agrees with a fundamental correlation length $\xi(p)$, defined, e.g., via the decay rate of the connectivity or the two-point function. Hence, if these correlation lengths exhibit critical scaling, the critical index ν obeys the bound $\nu \geq 2/d$. A Mandelbrot aerogel, when viewed in a certain light, is a percolation problem defined on a background of (fractal) dimension $d_H = 2 - |\log Q/\log N|$†. Hence if we are in the sol phase with $Q \geq Q_c$, one might expect that as $p \uparrow P_G(Q)$, ξ_1 could enjoy a bound similar to the one in equation (1.7*a*) with d replaced by d_H:

$$\xi_1 \geq \text{constant}\,|p - P_G|^{-2/d_H}. \tag{1.7b}$$

It turns out that this is indeed the case; this is the subject of theorem 4.2. As a corollary, we obtain the result that if a critical index ν exists for the correlation length in the Mandelbrot aerogels then

$$\nu \geq 2/d_H. \tag{1.8}$$

2. The phase diagram

Let us begin with a preliminary bit of notation. We will assume that some value $N \geq 2$ has been selected for once and all. Let us denote by $A_n \subset [0, 1]^2$ a configuration which might be observed after n iterations of the Mandelbrot process; explictly, A_n consists of an occupied subset of the tiling of the unit square by N^{2n} square tiles of scale N^{-n}.

We denote by $\mu_Q(-) \equiv \mu_{Q;n}(-)$ the usual Mandelbrot measure on the sets A_n and by $\mu_{Q,p;n}(-)$ the appropriate aerogel measure where, as explained in the introduction, the nth iteration occurs with the retention probability p. We shall denote by Θ_n the collection of configurations A_n in which an edge-connected cluster of occupied cells contacts both the left boundary $[\{0\} \times [0, 1])$ and the right boundary $(\{1\} \times [0, 1])$ of the unit square. Equation (1.3) may thus be succinctly expressed by

$$\theta_n(Q, p) = \mu_{Q,p;n}(\Theta_n) \tag{2.1a}$$

while we will now define

$$\theta_n(Q) = \mu_Q(\Theta_n) \equiv \mu_{Q,Q;n}(\Theta_n) \tag{2.1b}$$

so that the relevant measure is indicated by the argument of θ_n.

We start with an elementary result concerning the existence of various limiting θ's:

Proposition 2.1. For all Q,

$$\lim_{n\to\infty} \theta_n(Q)$$

exists and is (by definition) equal to $\theta_\infty(Q)$. For any (Q, p) with $Q < 1$, if $\underline{\theta}(q, p)$ is zero, then the full sequence $(\theta_n(Q, p))$ converges to zero.

† Although the background is itself a (multiscale) disordered medium, this fact will play no role in the correlation length bounds derived in this work. In this context, the additional disorder serves only to provide a minor annoyance which we dispense with in lemma 3.1.

Remark. The proofs of the above statements can be taken almost without modification from what is currently in the literature. However, to keep this note self-contained, we will provide the complete arguments.

Proof. The quantities $\theta_n(Q)$ comprise a monotone decreasing sequence and hence converge. By definition

$$\theta_\infty(Q)=\mu_q\Big(\bigcap_n \Theta_n\Big)=\lim_{n\to\infty}\theta_n(Q).$$

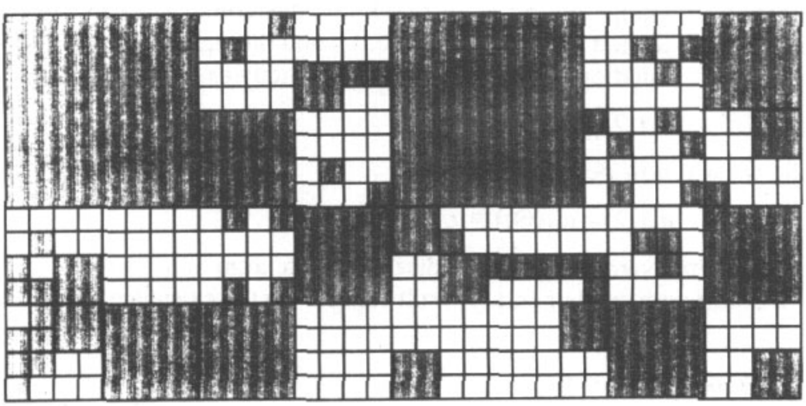
Let us now turn attention to the more substantial issue.

Suppose, then, that $\theta_n(Q,p)$ is small for some very large value of n. Let us consider two copies of the process placed side by side, so that the action is now taking place on $[0,2]\times[0,1]$. Consider the event, $\boldsymbol{B}_{2,n}$, that the top boundary of this rectangle ($[0,2]\times\{1\}$) has been disconnected from the bottom ($[0,2]\times\{0\}$) (figure 2). We claim that if n is sufficiently large, the fact that $\theta_n(Q,p)\ll 1$ implies that $\mathrm{Prob}(\boldsymbol{B}_{2,n})=\mu_{Q,p;n}(\boldsymbol{B}_{2,n})$ is close to one†.

Notice that the event Θ_n^c implies that there is a top–bottom crossing of $[0,1]^2$ by a crack of vacancy. Let us now consider the rotation of this event by 90° so that the vacant crossing goes from left to right. We divide the right boundary of $[0,1]^2$ into G equally sized segments where G is most easily visualized as N^k and the relevant k should be regarded as small compared to n.

Consider the event $\boldsymbol{B}^{[w]}_{1,n}$, $w=1,2,\ldots,G$, that a vacant crack crosses from the left boundary of $[0,1]^2$ and goes into (perhaps even subsuming) the wth segment on the right. A standard application of the Harris-FKG inequality shows that for some w^*,

$$\mathrm{Prob}(\boldsymbol{B}^{[w^*]}_{1,n})\geqslant 1-[1-\theta_n(Q,p)]^{1/G}. \tag{2.2}$$



† A similar assertion was made in [11] where, in this context, it was stated that an analogue of the RSW lemma [16, 17] holds for the Mandelbrot process. Although both of these facts are indeed correct, the derivation of the RSW lemma for Mandelbrot percolation found in the appendix of [11] is in error. A correct (albeit rather complicated) generalization of the RSW lemma which does work for these systems can be found in [18]. However, for the Mandelbrot-type systems, it is quite easy, following [19], to obtain long-way 'crossings of vacancy' by pasting together short ones. This is the route we will follow here. Two of the authors (C and C) would like to thank R Meester for pointing out the difficulty in our appendix.

Now, let us imagine that in the $[0,2]\times[0,1]$ system, the event $\boldsymbol{B}^{[w^*]}_{1,n}$ and its reflection across the line $x=1$ have both occurred. This gives us two cracks, each crossing its respective square, which are almost touching at the midline. These cracks will connect to form a single large crack if any of the pairs of adjacent squares (one just to the left and one just to the right of the line $x=1$) that include all of the w^*th segment within their boundaries is vacant (i.e. both squares in the pair must be vacant). The probability that this happens at least once is $1-(1-(1-Q)^2)^{n-k}$. The three above-mentioned events are positively correlated and together produce the event $\boldsymbol{B}_{2,n}$. Hence

$$\operatorname{Prob}(\boldsymbol{B}_{2,n})\geqslant(1-[\theta_n(Q,p)]^{1/G})^2[1-(1-(1-Q)^2)^{n-k}]. \tag{2.3}$$

It is seen in equation (2.3) that as θ_n gets small, one is permitted to use large values of G and hence large values of $n-k$. This drives the estimate on $\operatorname{Prob}(\boldsymbol{B}_{2,n})$ to one.

Now let m be large compared with some n for which $\operatorname{Prob}(\boldsymbol{B}_{2,n})$ is close to one. We will temporarily consider the unit scale to be the smallest squares and work our way upward to the full $N^m\times N^m$ lattice: We first 'vacate' unit squares with probability $(1-p)$, then group the squares into blocks of scale N, which (regardless of the interior pattern) become 'vacated' with probability $(1-Q)$, etc. Having done this n times, we now have $N^{2(m-n)}$ independent arenas where events such as Θ_n may succeed or fail. At this point, we may assume the worst case scenario (from the perspective the cracks) that between the nth and mth stages of this process none of the large-scale vacancy events occur. Even so, the $\boldsymbol{B}_{2,n}$ events enjoy a rescaling-type lemma (see, for example [14]) which demonstrates that the $\operatorname{Prob}(\boldsymbol{B}_{2,n+k})$ get driven to unity rapidly with k, and hence that $\theta_m\to 0$. Indeed, this occurs exponentially fast in the size of the block as measured in the above-mentioned unit scale. □

From the above arguments, we see that whenever any individual $\theta_n(Q,p)$ is small, the whole sequence gets driven to zero. This gives us the following:

Corollary. $\underline{\theta}(Q,p)$ is discontinuous across the phase boundary.

Proof. For $Q<1$, this statement follows immediately from the preceding derivation. Along the line $Q=1$, which contains the endpoint of the phase boundary, $(1,p_c)$, the situation is well understood: here the problem amounts to site percolation orchestrated in larger and larger boxes. The classic results of [16], [17], etc. (see [20]) tell us that $\theta_n(1,p)\to 0$ if $p<p_c$, $\theta_n(1,p)\to 1$ if $p>p_c$ and that $\theta_n(1,p_c)$ is uniformly bounded away from 0 and 1. □

We now tend to the central business of this section.

Proof of theorem 1.1. (1) and (4): For all p, it is clear that

$$\theta_{n-1}(Q)\geqslant\theta_n(Q,p) \tag{2.4a}$$

while for $p\geqslant Q$ we have

$$\theta_n(Q,p)\geqslant\theta_n(Q). \tag{2.4b}$$

Hence the limit exists and is zero when $Q<Q_c$, while if $p\geqslant Q$, the $\theta_n(Q,p)$ converge to $\theta_\infty(Q)$.

(2) Suppose $p<p_c$. We may, as in the proof of the preceding proposition, start on an $N^n\times N^n$ lattice and work our way outwards. The first step of the process looks like

ordinary site percolation, and hence, since $p<p_c$, the event Θ_n is immediately destroyed with probability tending rapidly to 1.

For $p=p_c$ and $Q<1$, we need consider only the first two steps of this outward-going process. Indeed, we may consider instead the following related percolation problem on the $N^n \times N^n$ sized square: Grouping the sites into $N \times N$ blocks, we will declare that any site is 'absent' if it is lost on the first step. However, those sites on the lower left corners of their respective blocks (and only those ones) will, in addition, be deemed absent if the whole block is lost on the second stage. Otherwise, sites will be deemed 'present'. Thus, in this problem, N^2-1 sits in each block are present with probability p, but the corner one only has density $p(1-Q)$. This is a classic setup for 'filing Kesten Ch. 10'; explicitly, if $Q<1$, this system is subcritical at $p=p_c$ [20]. See also the modern version of such arguments in [21]. Hence, for these parameter values, the probability of a connected 'present' crossing tends rapidly to zero with n, and the lack of such a crossing obviously implies that Θ_n is destroyed in the (first two stages of the) aerogel system.

(3) We will again construct a system for comparison; this time, it will be a large-N Mandelbrot percolation process. It is known that as N gets large, the threshold values $Q_c(N)$ approach p_c, the threshold of ordinary site percolation [22]. Suppose that $p>p_c$. Let us find a K such that $Q_c(M)<p$ whenever $M \geq K$. Let k denote the smallest integer for which $N^k \geq K$. Our comparison will have block groups of scale N^k. We may again envision working outward: first, at density p, we do the percolation step. Then, the next k steps of the aerogel process amount to a single step in the comparison process. We say that the big block is lost unless all k scales in the aerogel problem are completely retained. The probability of this is no less than Q^{kN^2} which, by insisting Q is close to unity, may now be assumed to exceed p. This comparison automatically establishes a subsequence of $\theta_n(Q,p)$'s, occurring each kth iteration, that are bounded below by the limiting $\theta_\infty(p)$ of the Mandelbrot process with N^{2k} subdivisions. Evidently, by the (contrapositive of) proposition 2.1, this establishes (3). In fact, it is not difficult to see that the *entire* sequence is bounded below by p^K times the above-mentioned limiting θ.

We make a few final observations which constitute a proof of statement (5): The existence of a $P_G(Q)$ is a tautology; the monotone properties of such a function follow from FKG-type considerations. It is also clear from items (2) and (3) that

$$\lim_{Q\to 1} P_G(Q)=p_c \tag{2.5}$$

and it has already been observed that $P_G(1)=p_c$. □

Remarks. The only qualitative features that we have failed to establish are (a) whether or not $P_G(Q)$ *is strictly* monotone; (b) where the phase boundary separates from the line $Q=Q_c$ (i.e. what is $P_G(Q_c)$?); and (c) the existence and behaviour of a limiting $\theta(Q,p)$. Our comments and speculations are as follows: (a) P_G is most likely to be a strictly monotone function; (b) there is no compelling reason to believe that $P_G(Q_c)=Q_c$ (this is deliberately obscure in figure 1); and, along these lines (c) we have some reason to believe that inside the phase boundary, $\theta(Q,p)=\theta(Q)$. The evidence supporting such an assertion is statement (4) of theorem 1.1 (i.e. that this occurs in the triangle $p \geq Q \geq Q_c$) and the fact that this is precisely what happens along the line $Q=1$. We do not have any intuition about the continuity of $\theta(Q,p)$—should it exist—as the phase boundary is approached from the gel phase. On the one hand, as

$Q \downarrow Q_c$ when $p > Q_c$, the system behaves like ordinary Mandelbrot percolation and hence θ is upper semicontinuous [11]. However, when $p < Q$, the argument proving semicontinuity fails and this may, perhaps, be more than a technicality. Indeed, as noted in the corollary to propositon 2.1, $\theta(1, p)$ is discontinuous as the pase boundary is approached from either side. Thus the question of continuity as $p \downarrow P_G(Q)$ remains open.

3. The correlation length

We will assume throughout this section that $Q > Q_c$ and $p < P_G(Q)$. Since, at present, we are dealing with a sequence of models each of which possesses only finitely many degrees of freedon, the correct choice of a correlation length is not readily apparent. Here we will propose two candidates for the correlation length of Mandelbrot aerogels and then demonstrate the scaling equivalence of these quantities: explictly, we will show that these quantities diverge as $p \uparrow P_G(Q)$, and they undergo these divergences in such a way that the ratio of their logarithms tends to unity.

Correlation length 1. Consider, as in proposition 2.1, a block of several independent copies of the Mandelbrot aerogel process. This time, it will prove convenient to use a 3×1 box, i.e. $[0, 3] \times [0, 1]$. Denote, as before, by $\boldsymbol{B}_{3,n}$ the event that the top of the box has been disconnected from the bottom. Let δ be a reasonably small number of order unity—e.g. any $\delta < \frac{15}{16}$ is sufficient. If $p < P_G(Q)$, we know that

$$\lim_{n\to\infty} \mathrm{Prob}(\boldsymbol{B}_{3,n}) = 1.$$

Let n^* denote the smallest integer such that

$$\mathrm{Prob}(\boldsymbol{B}_{3,n^*}) \geqslant 1 - \delta. \tag{3.1}$$

Our first correlation length is determined by this n^*: It is simply the linear dimension of the box, measured on the scale of the (current) lattice spacing. We define

$$\xi_1 = N^{n^*}. \tag{3.2}$$

Remark. The crucial feature of the condition $\mathrm{Prob}(\boldsymbol{B}_{3,n}) > 1 - \delta$ is that it is the hypothesis of a rescaling lemma obeyed by the quantity $\mathrm{Prob}(\boldsymbol{B}_{3,n})$. Such a rescaling lemma leads, in turn, to the consequences observed in the proof of proposition 2.1. It follows immediately that if $p \geqslant P_G$, then $\mathrm{Prob}(\boldsymbol{B}_{3,n}) \leqslant 1 - \delta$ for all n, and hence that n^* is infinite. Divergence of the length scale N^{n^*} as $p \uparrow P_G$ is easily demonstrated by continuity. An official proof of these assertions will appear in theorem 3.5.

Correlation length 2. Let us temporarily regard the nth stage aerogel as an $N^n \times N^n$ grid, most of which is vacant. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are points in $[0, 1]^2$, we will denote by $\tau_{\boldsymbol{a},\boldsymbol{b}}^{[n]}$ the probability that, in the nth stage aerogel, the cell (or cells) containing $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the same connected cluster. For present purposes, the case in which $\boldsymbol{a}$ and $\boldsymbol{b}$ differ only in a single coordinate is of sufficient generality. We will give this connectivity a special designation:

$$\tau_{\boldsymbol{a};s}^{[n]} \equiv \tau_{\boldsymbol{a},\boldsymbol{a}+(s,0)}^{[n]}. \tag{3.3}$$

Our second correlation length is the rate at which these probabilities tend to zero, measured in units of the cell size:

$$\tau^{[n]}_{a;s} \sim \exp[-sN^n/\xi_2]. \tag{3.4}$$

Of course later we are going to have to be a bit more precise, but for now we will write down the definition

$$-\frac{1}{\xi_2} = \frac{1}{s} \lim_{n\to\infty} N^{-n} \log \tau^{[n]}_{a;s}. \tag{3.5}$$

The existence of the above limit independent of s and $\boldsymbol{a}$ will be the subject of lemma 3.1.

We are now prepared to establish the existence, equivalence and critical divergence of these length scales. Our first goal is the result:

Lemma 3.1. For any $\boldsymbol{a} \in (0,1)^2$ and $s \in (-1, +1)$ with $\boldsymbol{a} + (s, 0) \in (0,1)^2$, the limit

$$\frac{1}{s} \lim_{n\to\infty} \frac{\log \tau^{[n]}_{a;s}}{N^n}$$

exists independent of $\boldsymbol{a}$ and s.

Our starting point will be a slightly different type of correlation which is intermediate between the connectivities leading to ξ_1 and ξ_2. We consider three copies of the nth stage aerogel placed alongside one another. If $\boldsymbol{a}$ is one of the $3N^n$ sites on the top ($[0,3] \times \{1\}$) and $\boldsymbol{b}$ is a site on the bottom, we define $G^{(n)}_{a,b}$ to be the probability that $\boldsymbol{a}$ is in the connected cluster of b and

$$G^{(n)} \equiv \max_{a,b} G^{(n)}_{a,b}. \tag{3.6}$$

The following is readily established:

Proposition 3.2. The limit

$$-\alpha \equiv \lim_{n\to\infty} \frac{\log G^{(n)}}{N^n}$$

exists. Furthermore, for any finite n

$$G^{(n)} \leqslant \frac{1}{Q^{3N}} \mathrm{e}^{-\alpha N^n}.$$

Proof. Suppose, for the nth stage problem, that $\boldsymbol{a}^*$ and $\boldsymbol{b}^*$ are the sites which maximize $G^{(n)}_{a,b}$:

$$G^{(n)}_{a^*,b^*} = G^{(n)}. \tag{3.7}$$

If N scaled-down translations and reflections of the event leading to $G^{(n)}_{a^*,b^*}$ are pasted together, one above the other, then there is a scaled-down connection in the $N \times 3$ box (figure 3(a)). This almost suffices to produce a connection of the desired type in the ($n+1$)th stage aerogel (in the left part of the scaled down $N \times 3N$ box). What is lacking is the assurance that this connection does not get 'blocked out' on (what is

Figure 3. (*a*) $N=4$ copies of the event $G^{(n)}_{a^*,b^*}$; (*b*) construction of $\mathcal{T}^{(n)}_{?}$.

currently) the first stage. However, this difficulty can be prevented by a 'plating factor' at the meagre cost of Q^{3N}. Thus one obtains, for particular sites $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$G^{(n+1)}_{A,B} \geqslant Q^{3N}[G^{(n)}]^N \tag{3.8}$$

so that

$$G^{(n+1)} \geqslant Q^{3N}[G^{(n)}]^N. \tag{3.9}$$

After k iterations of the above reasoning

$$G^{(n+k)} \geqslant Q^{V_k}[G^{(n)}]^{N^k} \tag{3.10}$$

where $V_k = 3N[N^k - 1]/[N-1] \leqslant 3N^{k+1}$. Equation (3.10) leads to a subadditive inequality that easily implies the desired results. □

Now let us consider the lattice $\mathbb{Z}^2$ where each site has been endowed with an independent copy of an nth stage aerogel at parameter values p and Q. We will assume, for simplicity that N is odd. Notice that this model is $\mathbb{Z}^2$ translation invariant (or, from the perspective of the unit cells, invariant under translations whose x_1 and x_2 components are integer multiples of N^n).

For x and y in $\mathbb{Z}^2$, we define $\mathcal{T}^{(n)}_{x,y}$ to be the event that there is a connection between the centers of the squares located at x and y and

$$T^{(n)}_{x,y} = \mathrm{Prob}(\mathcal{T}^{(n)}_{x,y}). \tag{3.11}$$

If x and y differ in only a single coordinate, we will abbreviate our notation: e.g. if

$x=0$ and $y=(r,0)$, we will call the corresponding probability $T_r^{(n)}$. The following proposition is readily verified:

Proposition 3.3. The limit

$$\lim_{r\to\infty} \frac{\log T_r^{(n)}}{r}$$

exists and, denoting this limit by $-\kappa_n$, the limit

$$\mu \equiv \lim_{n\to\infty} \frac{\kappa_n}{N^n}$$

also exists. Furthermore

$$T_r^{(n)} \leqslant \mathrm{e}^{-\kappa_n r}$$

and

$$\mu \leqslant \frac{\kappa_n}{N^n}.$$

Proof. The T functions are manifestly log superadditive, that is

$$T_{r+s}^{(n)} \geqslant T_r^{(n)} T_s^{(n)} \tag{3.12}$$

which immediately gives us our first result along with the corresponding *a priori* estimate. Furthermore, it is not hard to see that

$$T_{Nr}^{(n)} \geqslant T_r^{(n+1)} \tag{3.13}$$

from which the other results follow. □

We now begin the final leg of our demonstration of the equivalence of length scales.

Proposition 3.4. The quantities α and μ are equal.

Proof. We start with the straightforward result $\alpha \geqslant \mu$. Indeed, pick any n and locate the $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $G_{\boldsymbol{a},\boldsymbol{b}}^{(n)} = G^{(n)}$. By putting together $r-2$ translations and reflections of the event of a connection between points $\boldsymbol{a}$ and $\boldsymbol{b}$, we have, modulo contact at the end points, produced the event $\mathcal{T}_r^{(n)}$ (figure 3(*b*)).

Thus, we may write

$$T_r^{(n)} \geqslant V^2 [G^{(n)}]^{r-2} \tag{3.15a}$$

where, if hard pressed, we can pay the absurdly inflated price

$$V \geqslant p^{N^n} Q^{N^{n-1}} \dots Q^N \tag{3.15b}$$

to ensure the perfect status of the blocks at the beginning and end.

From equation (3.15*a*), we readily obtain

$$\kappa_n \leqslant |\log G^{(n)}| \tag{3.16}$$

and hence

$$\mu \leqslant \alpha. \tag{3.17}$$

As a preliminary step in the proof of the opposite inequality, we introduce the connectivity event $\mathcal{T}_r^{(n)\|}$ which means that there is a connection between $(0, 0)$ and $(r, 0)$ via a path which lies in the strip $0 \leq x_1 \leq r$†. We denote by $T_r^{(n)\|}$ the corresponding probability. It is completely straightforward to show that the limit of $-r^{-1} \log T_r^{(n)\|}$ exists. It is, in fact, relatively easy to show that this limit is precisely κ_n. Proofs of this sort of thing have appeared several times before in the literature [14]. However, for the sake of completeness, we will offer the following slight variant of the standard fare.

Denote by $\mathcal{T}_r^{(n);k} \subset \mathcal{T}_r^{(n)}$ the event of the desired connection taking place in the region $-k \leq x_1 \leq r+k$, with the usual notation for the corresponding probability. Let $q \geq 1$ and observe that while

$$T_{2qm}^{(n)\|} \leq T_{2qm}^{(n)} \tag{3.18}$$

we also have that

$$T_{2qm}^{(n)\|} \geq [T_m^{(n)\|} T_m^{(n);m} T_m^{(n);2m} \ldots T_m^{(n);qm}]^2. \tag{3.19}$$

Taking logs, etc, we get

$$\kappa_n \leq \lim_{r\to\infty} \frac{-\log T^{(n)\|}}{r} \leq \lim_{q\to\infty} \frac{1}{q} \sum_{j=0}^{q} \frac{-\log T^{(n);jm}}{m}. \tag{3.20}$$

Obviously the right hand side converges to $-m^{-1} \log T_m^{(n)}$, after which we can calmly take the $m \to \infty$ limit.

Finally, we will introduce one more correlation function, namely $T_r^{(n)\square}$ that, in addition to the restrictions which define $T_r^{(n)\|}$, has the further requirement that the connection takes place in the region $-r/2 \leq x_2 \leq +r/2$. The above correlation function has leading-order asymptotic behaviour identical to all the others; we will omit an explicit proof of this since similar topics have been adequately treated elsewhere [23].

In the next few paragraphs, we will use a coarse-graining argument to show that whenever the event $\mathcal{T}_r^{(n)}$ occurs, the connecting path is not teribly long. Let us denote by $\mathcal{K}_0^{(n)}$ the sites of $\mathbb{Z}^2$ which are connected to the origin by an underlying path of connected cells and by $|\mathcal{K}_0^{(n)}|$ the number of sites in $\mathcal{K}_0^{(n)}$. Observe that the sites in $\mathcal{K}_0^{(n)}$ themselves form a connected cluster. We claim that for $D \geq 1$

$$\text{Prob}(|\mathcal{K}_0^{(n)}| > D) \leq (\text{constant})\, e^{-\sigma_n D} \tag{3.21}$$

with $\sigma_n > 0$ if $\alpha > 0$ and n is sufficiently large. (If $\alpha = 0$, the equality of α and μ is assured by equation (3.17).)

To prove (3.21), consider a block of the nth stage aerogels organized into a 3×3 square block. We will say that there has been a *visitation* whenever there is a connection between a site in the inner square and the outer boundary of the block. Denote by $V^{(n)}$ the probability of a visitation. Note that a visitation is prevented if the four 3×1 rectangles surrounding the central square each have a long-way *vacant* crossing. Thus

$$V^{(n)} \leq 4G^{(n)}. \tag{3.22}$$

Consider, now, the event that $|\mathcal{K}_0^{(n)}| > D$. It is clear that for D big enough, the existence of such a connection necessarily implies the occurrence of some D distinct visitation events. Furthermore, the D 'sites' that get visited form a connected

† Since N and r are not necessarily commensurate, it may be that some of the relevant occupied cells are partially outside this strip.

cluster—although not necessarily a connected path. Unfortunately, these visitations events are not all independent. However, it is straightforward to show, as in the classic arguments of Russo [16], that at least $\frac{1}{9}$ of the events are mutually independent.

Indeed, if we tile the lattice with our 3×3 blocks, it is clear that any visitation events to the centers of distinct blocks are mutually independent. There are nine different ways that the lattice can be tiled; in at least one of these tilings, out of the total of D possible visitation events, as many as $D/9$ occur at the centres of the blocks. Hence, we may estimate

$$\mathrm{Prob}(|\mathcal{K}_0^{(n)}|>D \leqslant \sum_{K\geqslant D} \mathcal{V}_K[4G^{(n)}]^{K/9} \tag{3.23}$$

where $\mathcal{V}_D$ is the number of distinct connected clusters of size D that contain the origin.

It is well known that, to leading order, $\mathcal{V}_D$ grows exponentially fast:

$$\mathcal{V}_D \sim \mathrm{e}^{cD}. \tag{3.24}$$

Since we are assuming that α is positive, we will take n large enough to ensure that

$$\mathrm{e}^{-\sigma_n} \equiv \mathrm{e}^{c}[4G^{(n)}]^{1/9}<1. \tag{3.25}$$

Now, summing over all $K\geqslant D$, we obtain the claim made in equation (3.21).

With the above in mind, we may now take 'D' $=\Delta r$ with Δ sufficiently large, and observe that the events $\mathcal{T}_r^{(n)}$ and $\mathcal{T}_r^{(n)} \cap \{|\mathcal{K}_0^{(n)}|\leqslant \Delta r\}$ have essentially the same probability. Let us use this to construct the desired type of crossing.

We shall divide N^n into two scales: $n=k+(n-k)$ where both k and n are to be considered large. Looking at the square, with the origin of coordinates conveniently placed on the midpoint of the left side, let us allow, starting first at the smallest scales, $n-k$ iterations of the process. At this point, the probability of observing a left–right crossing which connects the midpoints is exactly $T_{N^k}^{(n-k)\square}$. (Recall the definition of $T_r^{(n)\square}$ from between equation (3.20) and (3.21).) However, we will make the somewhat more restrictive assumption that the event

$$\Lambda_0 = \mathcal{T}_{N^k}^{(n-k)\square} \cap \{|\mathcal{K}_0^{(n-k)}|\leqslant \Delta N^k\} \tag{3.26}$$

has occurred, where Δ has been chosen large enough to ensure

$$\mathrm{Prob}(\Lambda_0)\geqslant cT_{N^k}^{(n-k)\square} \tag{3.27}$$

for some number c which is of order unity. (Note that in arriving at equation (3.27) we cannot—and do not—make use of correlation inequalities. Indeed, (3.27) is easily obtained by inclusion–exclusion.)

Next, we will consider the event $\Lambda_1 \subset \Lambda_0$ which requires that in addition to Λ_0, on the $(n-k+1)$th iteration, none of the squares (of scale N^{n-k}) in 'the connected component of the origin' are lost. This event obviously has conditional probability $Q^{|\mathcal{K}_0|}\geqslant Q^{\Delta N^k}$. Continuing, we consider the event $\Lambda_2\subset\Lambda_1$, which further requires that none of the blocks of scale N^{n-k+1} that contain the previously mentioned blocks are lost on the next (i.e. $(n-k+2)$th) iteration. The conditional probability of Λ_2 is clearly Q raised to the power of the required number of blocks (i.e. the number of blocks which must be retained).

Obviously, this 'required number of blocks' lies somewhere between $|\mathcal{K}_0^{(n-k)}|$ and $|\mathcal{K}_0^{(n-k)}|/N^2$; furthermore, these blocks comprise a connected cluster in their own right. We claim that there is a number f strictly less than unity such that, no matter what shape the cluster assumes—provided that it is big enough—no more than $f|\mathcal{K}_0^{(n-k)}|$

blocks will be required. A proof of this (intuitively obvious) geometrical fact is relegated to the appendix. Explicitly then, we obtain the estimate

$$\mathrm{Prob}(\Lambda_2|\Lambda_1) \geqslant Q^{f\Delta N^k}. \tag{3.28}$$

In the same fashion as the second stage, we work our way up to the kth stage (i.e. the nth iteration). At the $(n-k+j)$th level, we define the event $\Lambda_j \subset \Lambda_{j-1}$, which requires that none of the blocks that were used to construct the event on the previous scale get lost on the present iteration. In other words, the entirety of $\mathcal{H}_0^{(n-k)}$ is retained in the current iteration. Until, perhaps, j is a few less than k, the conditional probability may be estimated:

$$\mathrm{Prob}(\Lambda_j|\Lambda_{j-1}) \geqslant Q^{f/\Delta N^k}. \tag{3.29}$$

It is also seen that the event Λ_k contains the desired sort of connection. Hence we finally arrive at

$$G^{(n)} \geqslant a[T_{N^k}^{(n-k)\square}]\,\mathrm{e}^{-AN^k} \tag{3.30}$$

where a and A are constants of order unity. Taking the log of both sides of equation (3.30), and defining $b=n-k$, we get, as n and k go to infinity with b fixed,

$$\alpha \leqslant \frac{A}{N^b} + \frac{\kappa_b}{N^b}. \tag{3.31}$$

The desired result is now established if we let $b \to \infty$. □

In a fashion quite similar to the preceding proof, we obtain:

Proof of Lemma 3.1. Let $\boldsymbol{a}$ and $\boldsymbol{a}+(s,0)$ be points in $(0,1)^2$, and suppose that in the nth stage aerogel there is a connection between these points. We may again write $n=b+k$ and regard the unit square as being tiled with kth stage aerogels destined to experience some disturbances at larger scales. From this perspective, the above-mentioned connection implies an event similar to $\mathcal{T}_{sN^b}^{(k)}$, the only differences being (a) the necessity of ensuring that contact has occurred between the *centres* of the relevant squares, (b) the fact that the separation between the relevant squares may differ from sN^b by one or two units, (c) interference from events of scale larger than N^k and (d) the fact that the connection is ostensibly hampered by the necessity of staying inside the (unit) square. For the purposes of a lower bound, items (c) and (d) can be ignored. We obtain

$$T_{sH_b}^{(k)} \geqslant \mathrm{e}^{-gN^k}\tau_{a;s}^{[n]}, \tag{3.32}$$

where the prefactor on the RHS accounts for item (a) and H_b is a number which differs from N^k by at most two.

Taking the appropriate limits in equation (3.32), we get the (easy) half of the desired result, namely

$$\liminf_{n\to\infty} \frac{|\log \tau_{a;s}^{[n]}|}{N^n} \geqslant \mu s. \tag{3.33}$$

For the second half, we will produce a connection between $\boldsymbol{a}$ and $\boldsymbol{a}+(s,0)$ by using the correlation functions $T_r^{(n)\square}$ and the plating technique of proposition 3.4. Let ε be some number no larger than s which is small enough to ensure that the points

$\boldsymbol{a}+(0, \pm\varepsilon)$ and $\boldsymbol{a}+(s, \pm\varepsilon)$ are safely inside $[0, 1]^2$. If we cannot get away with $\varepsilon = s$, for future convenience we will choose an ε which evenly divides s.

As before, let us first do k iterations starting at the smallest scale so that $[0, 1]^2$ is now covered with N^{2b} independent versions of the kth stage aerogel. The point $\boldsymbol{a}$ is in one of these squares, and (for b large) we may regard this square as being at the middle of the left side of a square of side ε. Exactly s/ε squares of this type are what separate $\boldsymbol{a}$ from its goal at $\boldsymbol{a}+(0, s)$. At this level of the process, we can accomplish most of what we need at a probabilistic cost of essentially $[T^{(k)\square}_{\varepsilon N^b}]^{s/\varepsilon}$; what is lacking are some terms that will guarantee that the cells containing $\boldsymbol{a}$ and $\boldsymbol{a}+(0, s)$ are actually part of the microscopic cluster. Furthermore, for the benefit of the rest of the argument below, we must take steps to ensure that the overall 'cluster' (the one consisting of squares of scale N^{-b}) is not overburdened with too many sites.

Given that all the above happens, we may work our way up through the remaining b iterations, guarding the cluster at each stage of the process as was done in the proof of proposition 3.4. When we are done, we have the estimate

$$\tau^{[n]}_{\boldsymbol{a};s} \geqslant (\text{constant})\, [T^{(k)\square}_{\varepsilon N^b}]^{s/\varepsilon}\, \mathrm{e}^{-g' N^k}\, \mathrm{e}^{-BN^b} \tag{3.34}$$

from which a bound complementary to the inequality in equation (3.33) emerges after a suitable limiting procedure. □

In light of what we have so far achieved, we may now dispense with all the previously defined symbols for asymptotic rates of decay of connections in favour of the unified notation $1/\xi_2$. We will finish this section with a (somewhate anticlimatic) proof of the central result for this section

Theorem 3.5. Let $Q > Q_c$ and $p < P_G(Q)$. The correlation lengths ξ_1 and ξ_2 are equivalent in the scaling sense. Explicitly, there are finite, positive constants C and K such that

$$\xi_2 \geqslant \frac{\xi_1}{C + K \log \xi_1}$$

while if δ has been carefully chosen to satisfy $(2N-1)^N \delta^N = \delta\, \mathrm{e}^{-N}$, then

$$\xi_1 \geqslant \xi_2.$$

Furthermore, these lengths actually diverge as the phase boundary is approached (i.e. as $p \uparrow P_G$).

Proof. Our starting point is the pair of inequalities

$$(3N^n)^2 G^{(n)} \geqslant 1 - \mathrm{Prob}(B_{3,n}) \geqslant G^{(n)}. \tag{3.35}$$

The second is self-explanatory; the first follows from subadditivity of the measure, summing over all possible endpoints of the connection. If $n = n^*$, the first (i.e. left) inequality in equation (3.35) and the a priori bound of proposition 3.2 immediately give

$$\xi_2 \geqslant \frac{\xi_1}{C + K \log \xi_1}. \tag{3.36}$$

The complementary bound $\xi_1 \geqslant \xi_2$ is proved by straightforward rescaling arguments that go back to [24]. Indeed, $\mathrm{Prob}(B_{3,n}) \geqslant 1 - \delta$ and a little cutting and pasting imply that $\mathrm{Prob}(\boldsymbol{B}_{3N,n}) \geq 1-(2N-1)\delta$, which yields (making no use of the available help on the largest scale)

$$\mathrm{Prob}(\boldsymbol{B}_{3,n+1}) \geq 1-(2N-1)^N\delta^N = 1-\delta\,\mathrm{e}^{-N}. \tag{3.37}$$

Then, after a total of k iterations,

$$\mathrm{Prob}(\boldsymbol{B}_{3,n+k}) \geq 1-\delta\,\mathrm{e}^{-N^k}. \tag{3.38}$$

Thus, in particular, we have

$$\mathrm{Prob}(\boldsymbol{B}_{3,n^*+k}) \geq 1-\delta\,\mathrm{e}^{-(N^{n^*+k})/\xi_1}. \tag{3.39}$$

Combining (3.39) with the second inequality in (3.35) and taking logs and limits, we obtain the desired bound.

It only remains to show that ξ_1 (and hence also ξ_2) diverges as $p \uparrow P_G$. To this end, observe that when $p \geq P_G$, $\forall n$, $\mathrm{Prob}(\boldsymbol{B}_{3,n}) < 1-\delta$; in essence this is just a recapitulation of proposition 2.1. Indeed, if this condition fails, e.g. at $p = P_G$ for some n, we would find ourselves in the sol phase. What is worse is that for larger n's, we would find $\mathrm{Prob}(\boldsymbol{B}_{3,n}) > 1-\delta$, and hence by continuity we could find an n and an $\varepsilon(n)$ such that $\mathrm{Prob}(\boldsymbol{B}_{3,n}) \geq 1-\delta$ at $P_G+\varepsilon$, in direct violation of the definition of $P_G(Q)$.

Now let us consider the situation when $p < P_G$. Since, at fixed Q, $\mathrm{Prob}(\boldsymbol{B}_{3,n})$ is a monotone function of p, it follows that n^* is also monotone in p. Hence

$$\lim_{p \uparrow P_G} n^*(p, Q)$$

either exists or is infinite. However, the preceding argument rules out a finite limit '$n^*(P_G)$'. In particular, finite-volume continuity would tell us that at $p = P_G$, $\mathrm{Prob}(\boldsymbol{B}_{3,n^*(P_G)})$ is itself at least as large as $1-\delta$ and we have just argued that this is impossible. □

4. Correlation length bounds

In the preceding section, it was agreed that the correlation length was determined (or defined) by the number of scales that are required before vacant cracks can be observed with a reasonable probability. The phrase 'reasonable probability' is, of course, somewhat arbitrary; to a certain extent, this is deliberate. Indeed, as should be clear from the orchestration of theorem 3.5, the choice of $\delta = [\mathrm{e}(2N-1)]^{-N/(N-1)}$ was for the aesthetic purpose of having no constants in the second inequality. It is, in fact, straightforward to show that the choice of $\delta' = \frac{1}{5}\sqrt{5}$ is sufficient in the sense that if for some n_0, $\mathrm{Prob}(\boldsymbol{B}_{3,n_0}) > 1-\delta'$, then $\mathrm{Prob}(\boldsymbol{B}_{3,n})$ will tend to zero exponentially fast in N^n. Of course, δ' isn't really any 'better' than the δ we used in theorem 3.5†. However, our proof below will exploit the fact that $\delta < \delta'$ (the result of a laborious calculation that we will not reproduce here) and the fact that the divergence established in theorem 3.5 holds with δ replaced by δ'. Explicitly, at any point on the

† It can be shown that the associated rate of decay will only differ from our ξ_1 by a constant factor. The [redacted]

phase boundary, $\mathrm{Prob}(B_{3,n})<1-\delta'$. Hence, focusing attention on some (p, Q) with $p<P_G(Q)$, in order that n be equal to $n^*(p, Q)$, the value of $\mathrm{Prob}(B_{3,n})$ must have changed by $\delta'-\delta$ (an amount of order unity) in the interval P_G-p.

In [12] and [15], useful bounds on the rates of change for probabilities of events defined in finite volumes were derived. The relevant version is:

Theorem 4.1 [12]. Let A denote an event which depends on the outcomes of a total of $|\Lambda|<\infty$ identical and independently distributed Bernoulli random variables of density $p\in(0, 1)$. Then

$$\left|\frac{\mathrm{d}\,\mathrm{Prob}(A)}{\mathrm{d}p}\right|\leqslant C(p)|\Lambda|^{1/2}$$

with $C(p)=[1/p(1-p)]^{1/2}$.

After a moment's thought, it is seen that the generalization needed here is to cases where 'Λ' itself is a random variable.

Proposition 4.2. Let $\mathbb{Z}_M$ denote the set of the first M positive integers (where, for simplicity, we assume $M<\infty$) and let η denote some fixed probability measure on the collection of subsets of $\mathbb{Z}_M$. If $\Lambda\subset\mathbb{Z}_M$, let $(X_j|\,j\in\Lambda)$ denote a collection of independent Bernoulli random variables where, for each j, $\mathrm{Prob}(C_j=1)=p$ for some $p\in(0, 1)$. Then, for any event A,

$$\left|\frac{\mathrm{d}\,\mathrm{Prob}(A)}{\mathrm{d}p}\right|\leqslant C(p)\mathrm{E}_\eta(|\Lambda|^{1/2})$$

where $|\Lambda|$ denotes the size of Λ and $C(p)$ is the constant in theorem 4.1.

Proof. We may express Prob(A) as a sum over conditional probabilities:

$$\mathrm{Prob}(A)=\sum_{\Lambda}\mathrm{Prob}(\Lambda)\mathrm{Prob}(A|\Lambda). \tag{4.1}$$

If $\omega\subset\Lambda$ is a configuration of the Bernoulli variables, we may write

$$\mathrm{Prob}(A|\Lambda)=\sum_{\omega}\mathrm{Prob}(\omega)\mathbb{1}_A(\omega) \tag{4.2}$$

where $\mathbb{1}_A(\omega)$ is the indicator for the event A.

We use the explicit formula for the Bernoulli distribution:

$$\mathrm{Prob}(\omega)=p^{|\omega|}(1-p)^{|\Lambda|-|\omega|} \tag{4.3}$$

(where $|\omega|\equiv\sum_i X_i$ is the size of ω) to obtain

$$\frac{\mathrm{d}\,\mathrm{Prob}(A|\Lambda)}{\mathrm{d}p}=\frac{1}{p(1-p)}\sum_\omega \mathrm{Prob}(\omega)(|\omega|-p|\Lambda|)\mathbb{1}_A(\omega). \tag{4.4}$$

Taking absolute values, we employ the triangle and Hölder inequalities to obtain

$$\left|\frac{\mathrm{d}\,\mathrm{Prob}(A|\Lambda)}{\mathrm{d}p}\right|\leqslant\frac{1}{[p(1-p)]^{1/2}}|\Lambda|^{1/2} \tag{4.5}$$

where the preceding calculation has been expedited by the observation that $p|\Lambda|$ is the conditional expectation of $|\omega|$. The desired result is immediately obtained by multiplying both sides of equation (4.5) by $\mathrm{Prob}(\Lambda)$ and summing over Λ. □

Remark. In cases when $M=\infty$, it is not particularly difficult to extend proposition 4.2 provided that all terms are well-defined.

Our princial result is a straightforward application of this proposition:

Theorem 4.3. Let $Q>Q_c$ and $p<P_G(Q)$. Then there is a positive, finite constant A such that

$$\xi_1(p,Q)\geqslant A|P_G-p|^{-2/d_H}$$

where

$$d_H=2-\frac{|\log Q|}{\log N}.$$

Remark. The interpretation of d_H as the Hausdorff dimension of the limit set of the Mandelbrot percolation process has been given in some detail in [11].

Proof. Let $n\gg1$. Let us use proposition 4.2 to estimate the rate of change of $\mathrm{Prob}(\boldsymbol{B}_{3,n})$ as a function of p. The relevant Bernoulli variables are, of course, the single-cell occupation events that take place against the background of (three independent copies of $(n-1)$ iterations of) the Mandelbrot process with parameters N and Q. Thus, the number of available places (i.e. '$|\Lambda|$') is distributed according to the sum of three independent branching processes that enjoy a mean of QN^2 progeny. Explicitly, if we denote these objects as Z_1^{n-1}, Z_2^{n-1} and Z_3^{n-1} respectively, it is seen that the number of available sites is distributed as $N^2(Z_1^{n-1}+Z_2^{n-1}+Z_3^{n-1})$. Hence

$$\left|\frac{\mathrm{d}\,\mathrm{Prob}(\boldsymbol{B}_{3,n})}{\mathrm{d}p}\right|\leqslant\frac{N}{[p(1-p)]^{1/2}}\mathsf{E}[(Z_1^{n-1}+Z_2^{n-1}+Z_3^{n-1})^{1/2}]. \tag{4.6}$$

As a bound, we will replace the average of $(Z_1^{n-1}+Z_2^{n-1}+Z_3^{n-1})^{1/2}$ with $\sqrt{3}[\mathrm{E}(Z_1^{n-1})]^{1/2}=\sqrt{3}(QN^2)^{(n-1)/2}$†.

Focusing our attention, for once and all, on some fixed region abut P_G, we may ignore the variation of the prefactor and rewrite equation (4.6) as

$$\left|\frac{\mathrm{d}\,\mathrm{Prob}(\boldsymbol{B}_{3,n})}{\mathrm{d}p}\right| \leqslant K[N^n]^{d_H/2} \tag{4.7}$$

with K a constant. But then, integrating from p to P_G (ignoring the variation of K with p), and using the fact discussed earlier that $\mathrm{Prob}(\boldsymbol{B}_{3,n})<1-\delta'$ along the phase boundary, we have

$$\mathrm{Prob}(\boldsymbol{B}_{3,n})\leqslant 1-\delta'+K[N^n]^{d_H/2}[P_G-p]. \tag{4.8}$$

Thus all ns that satisfy $K[K^n]^{d_H/2}[P_G-p]<\delta'-\delta$ must be smaller than $n^*(p)$. Evidently,

$$\xi_1\geqslant[(\delta'-\delta)/K]^{2/d_H}[P_G-p]^{-2/d_H} \tag{4.9}$$

as stated above □

Similar bounds, with logarithmic modifications, may be obtained for ξ_2 if we care to use the results of theorem 3.5. However, we will be satisfied with a concluding statement on the correlation length exponent:

Corollary. If the critical index ν exists, even in the sense of

$$\nu=\lim_{p\uparrow P_G}\left|\frac{\log\xi}{\log(P_G-p)}\right|,$$

then $\nu\geqslant 2/d_H$.

Appendix

Here we prove the geometrical result needed in the proof of proposition 3.4 (see equation (3.28)).

Theorem A.1. Let $K\subset\mathbb{Z}^2$ denote a connected set of size $|K|$ with $0\in K$. Consider a tiling of $\mathbb{Z}^2$ by squares of linear dimension $N\geqslant 2$. Let $|K'|$ denote the number of distinct tiles which have non-zero intersection with the cluster K. Then for all K with $|K|$ sufficiently large, there is a constant $f<1$ such that

$$|K'|\leqslant f|K|.$$

Remark. There is little doubt that this result— probably proved in greater generality and with better constants—has appeared somewhere in the literature. However, the authors have proved unequal to the task of divining the precise location of such a theorem.

† In our case, this final bound occurs without any significant loss since, for $n\gg 1$, the quantities Z_k^n have, in the exponential sense, sharply peaked distributions. One of the authors (LC) would like to thank T Liggett for an enjoyable afternoon discussing this topic.

Proof. Let us partition the tiling of $\mathbb{Z}^2$ (itself a realization of $\mathbb{Z}^2$) into nine independent sublattices which are labelled Green and 'coloured' $C_2, \ldots, C_8$ (figure 4). We will assume that K is of sufficient size to have non-zero overlap with at least two tiles from one of the sublattices. We will further assume, with no loss of generality, that the sublattice which contributes the most tiles to the covering of K is the Green one.

Let us consider, then, those Green tiles which intersect the cluster K. Each such tile is surrounded by an annulus composed of all the other coloured tiles. Furthermore, the annuli surrounding the distinct Green tiles are disjoint. We claim that in every annulus surrounding a Green tile which has been used in the covering, at least one of the tiles will contain more than $N/2$ sites of the cluster K.

Indeed, in order that some site in a given Green tile be connected to the 'rest of K' (e.g. another Green tile), it follows that K must include a connected path running between the inner and outer boundaries of the annulus. This path, in turn, must cross one of the four (overlapping) 3×1 rectangles, e.g. the rectangle consisting of tiles coloured C_4 C_6 and C_9. If this latter portion of the path connects the boxes of all three colours, it is seen that the middle box has been used on at least N occasions. If only one box is used, the result is equally obvious. Finally, if two boxes are used, at least one of them must contain a chain of half its breadth, otherwise the path will not reach across. With somewhat more precision, $(N+1)/2$ is the minimum requirement if N is odd and at least $N/2+1$ sites are necessary when N is even. To simplify the remainder of this proof, we will henceforth assume that N is even.

Let g, $c_2 \ldots, c_8$ denote the number of sites in K which end up in the Green, $C_2, \ldots, C_8$ coloured tiles, and let $\mathsf{G}, \mathsf{C}_2, \ldots, \mathsf{C}_8$ denote the number of tiles of the assorted colours used in the covering. Associated with each of the G Green tiles is a (possibly non-unique) constitutent of the appropriate annulus which has more (or as many) sites of K as does any other member of the annulus. Let α_k denote the fraction of times that the colour C_k is awarded this dubious distinction. Thus

$$\sum_k \alpha_k \mathsf{C}_k \geqslant \mathsf{G}. \tag{A.1}$$

Knowing that each of these 'winners' contain at least $N/2+1$ sites of K, it is seen that we may estimate

$$\mathsf{C}_j \leqslant \alpha_j \mathsf{C}_j + (c_j - [N/2+1]\alpha_j \mathsf{C}_j) = c_j - N/2\alpha_j \mathsf{C}_j \tag{A.2}$$

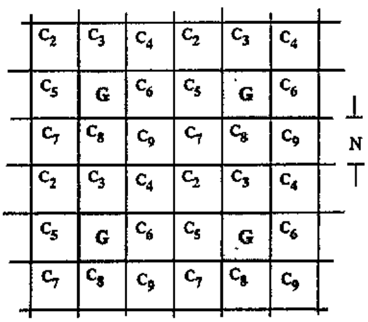


Figure 4. A tiling of $\mathbb{Z}_2$.

which follows from (crudely) bounding the number of 'non-winning tiles' by the number of sites in K which are in 'non-winning tiles'. Equation (A.2) may be supplemented with the obvious bound

$$\mathrm{G} \leqslant g. \tag{A.3}$$

Adding up the left and right hand sides of the above two equations, we obtain

$$|K'| \leqslant |K| - (N/2)\,\mathrm{G}. \tag{A.4}$$

Since $\mathrm{G} \geqslant (\frac{1}{9})|K|$, the desired result has been established with $f = [1 + N/18]^{-1}$. When N is odd, the constant f is given by the same formula as above with N replaced by $N-1$. □

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